Adaptive maximum-likelihood estimation of psychometric slope

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This approach follows Green [1] and requires identifiying the "sweet points", that is the points associated with the minimum expected variance of the slope estimate. For the slopes there are two such points, placed above and below the threshold. Following Green [1], one can compute them by treating the psychometric function as locally linear.
In our case the psychometric function is

$$P(x) = \lambda + (1 - 2\lambda)\Phi(x) \tag{1}$$

where

$$\Phi(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x - \mu}{\sqrt{2}\sigma}\right)\right] \tag{2}$$

Having an estimates of the parameters ($\hat{\mu}$, $\hat{\sigma}$, and $\hat{\lambda}$), one can select the stimulus which will minimize the expected variance of each one of them. Treating the psychometric function as locally linear (for a line $y = ax + b$ the variance of y is equal to a^2 times the variance of x), the expected variance of the estimate of $\hat{\sigma}$ can be computed as

$$Var(\hat{\sigma}) = \frac{P(x)\left[1 - P(x)\right]}{\left(\frac{d}{d\sigma}P(x)\right)^2} \tag{3}$$

Using the psychometric function defined in equation 1 and 2, one can compute the expected variance of the slope as

$$Var(\hat{\sigma}) = \frac{\pi\sigma^4 e^{\left(\frac{x-\mu}{\sigma}\right)^2}\left[1 - (1-2\lambda)^2 \text{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]}{2(1-2\lambda)^2(x-\mu)^2} \tag{4}$$

In fig. 1 I plotted the expected variance as a function of the probability predicted by the psychometric function (left panel) and as a function of the

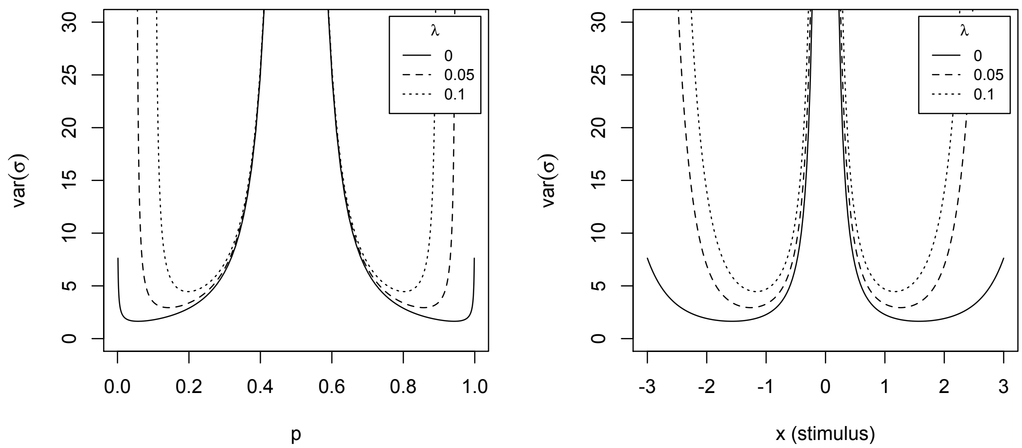

Figure 1: Expected variance of the estimated slope $\hat{\sigma}$, for different values of the parameter λ, plotted as a function of the probability predicted by the psychometric function on the left, and as a function of the stimulus on the right. (For this plot and the one in fig. 2 I used $\mu = 0$ and $\sigma = 1$.) The "sweet points" are the minima of the expected variance.

stimulus (right panel), for different levels of the parameter λ, which gives the probability of lapses, or random responses. With increasing values of λ, the "sweet points", that is the minima of the function giving the expected variance for the slope, are shifted toward the threshold ($\mu = 0$).

Hence a simple maximum-likelihood adaptive procedure could be to use the sweet points as next stimuli location. This would correspond to choosing stimuli that minimize the expected variance of the parameter estimates (rather than the expected entropy of the joint parameter probability density, as in Quest+ [2]). One could start with few random trials, then obtain an estimate of the parameters by maximum likelihood, and presents the following stimuli at the sweet points (updating the ML estimates after each trial). The expected variance of the parameter λ does not have a minimum (it keeps decrease as one moves away from the threshold), so to estimate it properly one could include some trials at very high stimulus value. The sweet point for the mean μ is the mean itself (see fig. 2), and one could present a fraction of stimuli at the estimated threshold.

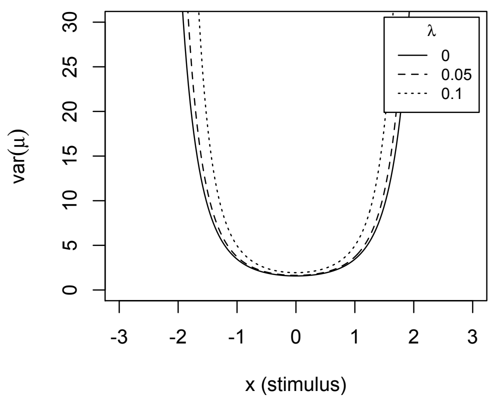

Figure 2: Expected variance of the estimated mean $\hat{\mu}$, for different values of the parameter λ.

In fig. 3 is shown 100 trials of such adaptive procedure: 20 random trials followed by a series of trials where the stimulus is placed at one (randomly) of the two sweet points, or at the mean (with probability 1/4). The probability of lapses is in this case taken into account, and estimated after each trial.This method is heuristic in some aspects (e.g. deciding which sweet point should be tested next) but has the advantage that can be modulated so as to improve specifically the estimates of the slopes (a the expenses of the other parameters).

References

[1] D. M. Green. Stimulus selection in adaptive psychophysical procedures. *The Journal of the Acoustical Society of America*, 87(6):2662–2674, jun 1990.

[2] A. B. Watson. QUEST+: A general multidimensional Bayesian adaptive psychometric method. *Journal of Vision*, 17(3):10, 2017.

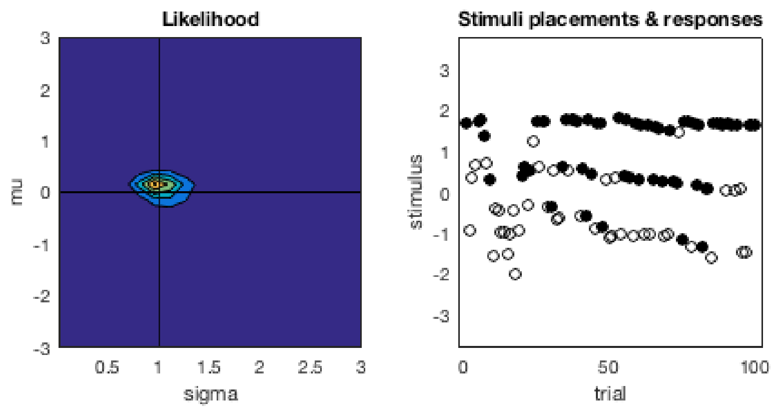

Figure 3: Adaptive maximum-likelihood estimation of a psychometric function. Left panel: likelihood density over the parameter after 100 trials. Right panel: stimuli placement, filled dots indicate '+' responses from a simulated observer with probability of lapsing $\lambda = 0.01$.