On Signal Detection Theory & Generalized Linear Models

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GLM formulation of equal-variance SDT models

Signal detection theory models relies on the assumption that in a perceptual detection task the information that is available to the observer can be modeled as a random variable S, which is drawn from either the *signal* distribution $f_S(s)$, if the signal is present, or the *noise* distribution $f_N(s)$, if it is absent. Often these distributions are assumed to be Gaussian, but they can modeled also with other probability distributions in specific applications.
If the variance of the two distributions is assumed to be equal in both signal and noise distribution, then the only difference between them is an additive constant, that is the d' or sensitivity index. In this case the signal distribution is also called *signal+noise* distribution, and is a normal distribution with mean $d' > 0$ and variance $\sigma^2 = 1$.

$$f_S(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(s-d')^2}{2}} \tag{1}$$

while the noise distribution is

$$f_N(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(s)^2}{2}} \tag{2}$$

The other parameter of the equal-variance SDT model is the *sensory criterion*, c. The observer responds that the signal was present whenever $s \geq c$, and will respond absent otherwise. The *optimal* detection criterion will depends on the prior probability of the signal (i.e. the proportion of trials where the signal is present). If the probability of the signal is p_s, then the optimal criterion c can be computed as

$$c = \frac{\log\left(\frac{1-p_s}{p_s}\right)}{d'} + \frac{d'}{2} \tag{3}$$

It is easy to verify that when $p_s = 0.5$ the ideal criterion becomes $c = \frac{d'}{2}$.
Note that the decision rule expressed in this way (a comparison with respect to the sensory criterion) is a simplification of the more general rule based on the log-likelihood ratio of the probability densities of the signal and noise distribution at the point s (which indicates the sensory information available to the observer on a given trial). According to the log-likelihood ratio decision rule the observer respond that the signal was present whenever

$$\log \frac{f_S(s)}{f_N(s)} \geq \log \frac{1-p_s}{p_s} \tag{4}$$

The parameters of the equal-variance SDT model are typically computed by their analytical maximum likelihood estimators, using the observed proportion of hits p_{H} and of false alarm p_{FA}

$$d' = \Phi^{-1}(p_{\text{H}}) - \Phi^{-1}(p_{\text{FA}}) \tag{5}$$

$$c = -\Phi^{-1}\left(p_{\mathrm{FA}}\right) \tag{6}$$

where Φ^{-1} is the inverse of the cumulative distribution function of the standard normal distribution, also known as *quantile* function (in `R` is implemented in the function `qnorm()`). The basic equation of a *probit* generalized linear model, for the same situation, could be expressed as

$$\Phi^{-1}\left(p_{yes}\right) = \beta_0 + \beta_1 X \tag{7}$$

where p_{yes} is the probability of the observer responding that the signal was present, and X is a variable that indicates the presence/absence of the signal as 1/0. The similarity with the SDT model is evident if we consider that, in the GLM, the probability of a hit or a false allarm correspond to p_{yes} when the signal is absent (that is $X = 0$) or present (that is $X = 1$), respectively.

$$c = -\Phi^{-1}\left(p_{\mathrm{FA}}\right) = -\beta_0 \tag{8}$$

$$d' = \Phi^{-1}\left(p_{\mathrm{H}}\right) - \Phi^{-1}\left(p_{\mathrm{FA}}\right) = \beta_0 + \beta_1 - \beta_0 = \beta_1 \tag{9}$$

The parameters of the GLM models can thus be mapped directly to the SDT parameters. By recognizing this identity it is possible to use existent statistical packages, such as `R`, to easily analyze complex design made of multiple conditions and interaction effects. It is also possible to extend the classical SDT model to include random effects.